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Sliding mode control in a bioreactor model

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This article deals with an example of nonlinear control systems and the interlacing between a biochemical system, the mathematical model and the constrains derived from the discrete implementation of a continuous control policy. The theory is developed on a simplified model of a bioreactor to be regulated, and the sliding mode control is presented as a robust control technique. The biological interpretation of the results derived from the mathematical model is pointed out, especially of those more closely involved with the implementation, as is the case of sample period, which seems to be very enough with respect to the minimum time needed for sample analysis.

KEY WORDS: bioreactor, nonlinear systems, sliding mode control

1. Introduction

Interdisciplinary resources mixing content from different topics which integrate real life problems are currently necessary in scientists and engineers education. This paper presents sliding mode control introducing the reader into the theory through a biochemical example by mixing maths, physics, biology and chemistry. The interdisciplinary nature of this project is of special interest for knowledge integration, which so often appears to be sealed in separate worlds in university students education.

By means of this example, some details of sliding mode control are introduced and studied more clearly. The discussion of the obtained results leads an excellent opportunity for a better understanding of the concepts.

The theory to be presented is developed through a real life problem: the control of a bioreactor which is presented as a nonlinear system to be regulated. Through the example the notions of dimensionless variables and parameters, and significant data will

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be discussed and the results, for instance, the real meaning of the variable time, will be interpreted.

It is worth emphasizing the nonlinear dynamic behaviour of biological systems. Linear control system design generally fails or shows very poor performance, even in a very small area of operation in the phase plane, and therefore further advanced control schemes have to be applied. Sliding mode control is the technique considered here since it provides very simple control policies and it is robust in the presence of plant parameter variations. It is based on ideal sliding dynamics defined by a sliding surface and a control policy in such a way that the sliding surface becomes an atractor for the dynamical system. See [1–4], where detailed background material can be readily found.

The bioreactor is a challenging control problem for several reasons. Although the task involves few variables and is easily simulated, its nonlinearity makes its control difficult. This problem satisfies the goals of relevance to real-life problems and is easy enough to be presented as an academic example. Moreover, improvements in bioreactor control techniques can result in significant savings for biochemical industries and in the enhancement of the productivity of large volume applications.

The outline of the paper reads as follows. After the introduction, the problem statement is presented in section 2. Section 3 is devoted to a brief introduction on sliding control theory, which leads to a control strategy for the bioreactor in section 4. Section 5 shows the discretization of the sliding control and introduces some simulations; final comments on application and a list of references close the work.

2. Problem statement

2.1. Model

The plant with which we are dealing is a continuously stirred tank reactor. The tank contains a liquid mixture: water, nutrients and biological cells. Feed substratum is introduced into the tank where the cells mix with the substratum while the culture volume is kept constant. The biological process carried out in the tank is described by the following states and parameters: cell concentration, X, substratum concentration, S, feed concentration of substratum, S_F, reactor volume, V, volumetric feed flow rate, F, specific growth rate, $\mu(S)$, specific substratum consumption rate, $\sigma(S)$, and real time t. The evolution of the process is described by

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{FX}{V} + \mu(S)X, \tag{1}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{F(S_F - S)}{V} - \sigma(S)X. \tag{2}$$

The reader is referred to the tutorial paper of Bastin and Van Impe [5] for an extensive introduction on the model and control issues for bioreactors.

2.2. System dynamics

The previous model is particularized to an inhibited substratum model, for which, as in [6], the normalized specific growth rate and the normalized substratum consumption rate are, respectively,

$$\frac{\mu(S)}{\mu(S_F)} = (1-N)\mathrm{e}^{N/\gamma}, \qquad \frac{\sigma(S)}{\sigma(S_F)} = (1-N)\mathrm{e}^{N/\gamma}\frac{1+\beta}{1+\beta-N}.$$

Thus, equations (1) and (2) become

$$\frac{\mathrm{d}C}{\mathrm{d}t^*} = -C\omega + C(1-N)\mathrm{e}^{N/\gamma}, \tag{3}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t^*} = -N\omega + C(1-N)\mathrm{e}^{N/\gamma}\frac{1+\beta}{1+\beta-N}, \tag{4}$$

where C is the normalized cell concentration $C = X/(Y(S_F)S_f)$, Y is the "yield coefficient" $Y(S) = \mu(S)/\sigma(S)$, N is the substratum conversion $(N = 1 - S/S_F)$ and t^* is a dimensionless time, which verifies

$$t^* = t\mu(S_F). \tag{5}$$

Roughly speaking, these equations model the mass balance between cells and substratum determining the evolution of the dynamical system. The input variable

$$\omega = \frac{F}{\mu(S_F)V}$$

is proportional to the flow running through the tank.

The system shows an autonomous growth model based on experimentation.

Some constraints must be considered. Cell concentration and substratum conversion belong to the interval [0, 1], $(C, N) \in [0,1] \times [0,1]$; the input variable is positive and less than or equal to 2, $\omega \in [0, 2]$. As in [6] the growth rate parameter γ is 0.48 and the nutrient inhibition parameter β equals 0.02.

The objective is a regulation problem; that is to say, achieving and maintaining a desired amount of cells or a substratum concentration by acting on the flow rate. In the language of dynamical systems, an appropriate flow rate must be designed in order to force the controlled system to have a previously stated equilibrium point which, in turn, should be stable.

3. Background

Equilibrium points play a significant role in regulation problems since the steady state is usually an equilibrium point.

Definition 1. $x = x^*$ is an equilibrium point for the system. Then

$$\dot{x} = F(x) \tag{6}$$

if and only if $F(x^*) = 0$.

Note that the trajectory $x(t) = x^*$ satisfies (6). Roughly speaking, this equilibrium point is stable if trajectories starting close to x^* go towards it.

There are several control methods for regulating a system. According to the goal of this paper, sliding mode control has been chosen here. This method is based on the two following main concepts:

- to define a surface, i.e., a relationship between state variables, in such a way that, if trajectories slide on this surface, a previously stated behaviour (for instance, reaching an equilibrium point) is achieved;
- to design an appropriate control law forcing this surface to be an attractor and a dynamically invariant set.

Let us consider a single input dynamical system given by

$$\dot{x} = f(x) + ug(x), \tag{7}$$

where $x \in U$, an open set of $\mathbb{R}^n$, f and g are smooth vector fields on U with $g(x) \neq 0$ everywhere, and $u : U \to \mathbb{R}$ is the control input.

Let Σ be a submanifold in U defined by a smooth function $s : U \to \mathbb{R}$, namely,

$$\Sigma = \bigl\{x \in U \mid s(x) = 0\bigr\}, \tag{8}$$

where $(\mathrm{grad} s)(x) \neq 0\ \forall x \in U$ and $\Sigma \cap U \neq \emptyset$ are assumed.

As for the input, let us take u defined by

$$u = \begin{cases} u^+(x) & \text{if } s(x) > 0, \\ u^-(x) & \text{if } s(x) < 0, \end{cases} \tag{9}$$

where both u^+ and u^- are smooth functions of x. There is no loss of generality in assuming $\langle \mathrm{grad} s, g\rangle > 0$.

Finally, let $\phi(x, t)$ be the trajectory of the dynamical system defined by (7), (8) and (9) with initial conditions $x(0) = x$. It is worth remarking that the former dynamical system is discontinuous on $H = 0$; thus, the standard results on differential equations do not apply. We will deal with this subject later; let us assume for the moment the existence and uniqueness of trajectories.

Definition 2. Σ is said to be a sliding surface for the dynamical system defined by (7), (8) and (9) if there exists θ, an open set in U containing Σ, in such a way that $\forall x \in \theta \setminus \Sigma$, one of the following conditions holds:

1. There exists a finite time $t_s > 0$ such that

$$s\bigl(\phi(x, t)\bigr) \neq 0,\ 0 \leqslant t < t_s \quad \text{and} \quad s\bigl(\phi(x, t)\bigr) = 0,\ t \geqslant t_s.$$

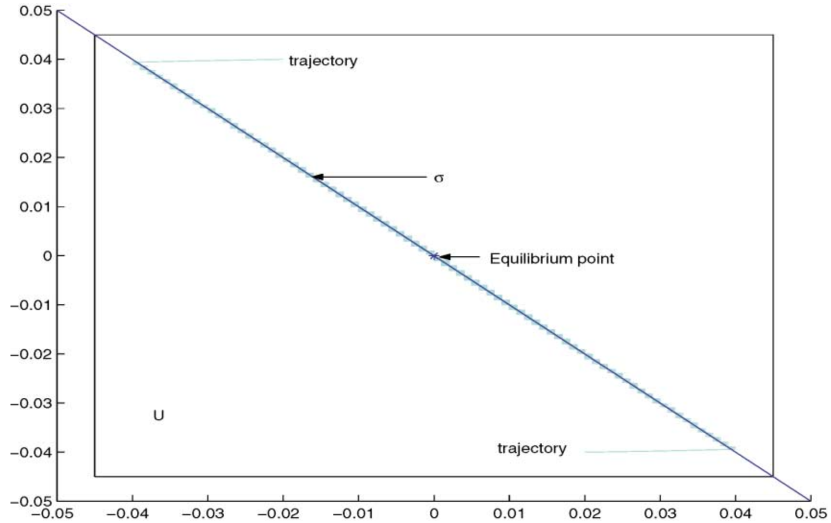

Figure 1. An example of sliding, first case.

2. There exist t_s and $\widehat{t_s}$, $0 < t_s < \widehat{t_s} < \infty$ such that

$$s\big(\phi(x,t)\big) \neq 0,\ 0 \leqslant t < t_s \quad \text{and} \quad s\big(\phi(x,t)\big) = 0,\ t_s \leqslant t < \widehat{t_s},$$

   and $\phi(x, \widehat{t_s}) \in \partial(\Sigma \cap U)$.

Roughly speaking, the trajectories starting in a neighbourhood of Σ must fall on Σ and remain there (case 1) or, should one escape, it must go through $\partial(\Sigma \cap U)$.

An example of case 1 is depicted in figure 1 where two trajectories, starting very close to Σ, fall on it and remain there converging to an equilibrium point of the ideal sliding dynamics. In figure 2 the behaviour described in case 2 is depicted. Two trajectories starting close to Σ fall on it and remain there in the open set $U \cap \Sigma$ and escape from Σ through $\partial(U \cap \Sigma)$.

As a first consequence of the definition, two questions arise, namely:

1. *Existence*. Which conditions on f, g, u and Σ, if any, guarantee that Σ be a sliding surface?

2. *Ideal sliding dynamics*. Note that the dynamics defined by (7), (8) and (9) do not consider Σ; however, if Σ is a sliding surface, it is dynamically invariant. Then the question is which vector field governs the system on Σ.

In the next section, both problems have been solved respectively.

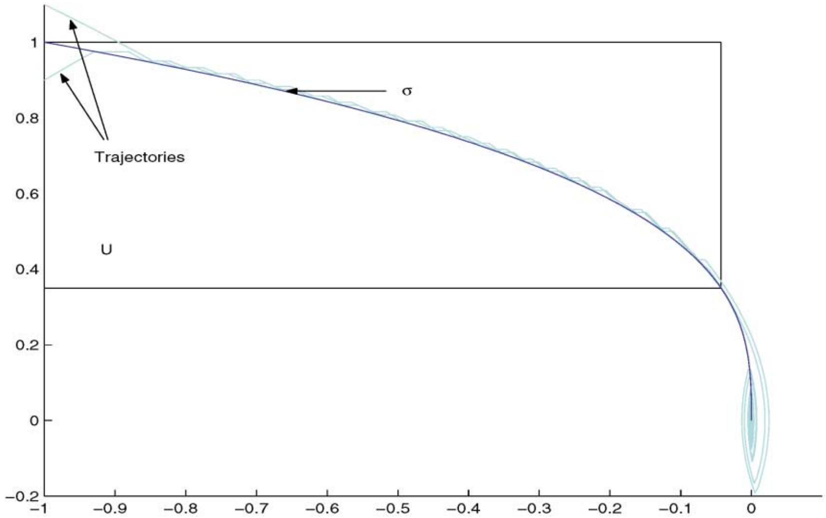

Figure 2. An example of sliding, second case.

3.1. Method of equivalent control and ideal sliding dynamics

Definition 3. Let us define equivalent control as the control law, $u_{\mathrm{eq}} : U \to \mathbb{R}$, which makes Σ an invariant manifold for the dynamical system defined in (7), that is to say, u_{eq} is such that the vector field $f + gu_{\mathrm{eq}}$ is tangent to Σ. This results in

$$\langle \mathrm{grad} s,\ f + gu_{\mathrm{eq}} \rangle = 0, \tag{10}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product, and thus,

$$u_{\mathrm{eq}} = -\frac{\langle \mathrm{grad} s,\ f \rangle}{\langle \mathrm{grad} s,\ g \rangle}. \tag{11}$$

As it is proved in [7], a paper by Filipov on differential equations with discontinuous right-hand side, ideal sliding dynamics, i.e., the dynamics on Σ, are governed by the vector field

$$f(x) + g(x)u_{\mathrm{eq}}(x).$$

Notice that a necessary condition for the existence of equivalent control is $\langle \mathrm{grad} s, g \rangle \neq 0$.

This equivalent control makes the sliding surface dynamically invariant. Hence, system trajectories reaching Σ slide on it.

Ideal sliding dynamics remains to be studied, particularly by computing possible equilibrium points and determining whether they are stable or not.

3.2. Control law and sliding motion

As far as existence is concerned, two results, depending on whether u^+ and u^- are fixed or not, are given in this subsection.

Proposition 1. Σ is a sliding surface for the dynamical system defined by (7), (8) and (9) if and only if there exists θ, a neighbourhood of Σ, such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} s\big(\phi(x,t)\big) < 0 & \text{if } s\big(\phi(x,t)\big) > 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t} s\big(\phi(x,t)\big) > 0 & \text{if } s\big(\phi(x,t)\big) < 0. \end{cases}$$

Remark. We consider derivatives of s along the trajectories of the vector field $f(x) + ug(x)$ for the values of u defined in (9). These conditions may also be written as

$$\begin{cases} \lim_{s\to 0^+} L_{f+gu^+} s(x) < 0, \\ \lim_{s\to 0^-} L_{f+gu^-} s(x) > 0, \end{cases} \tag{12}$$

where $L_{f+gu^+} s(x)$ denotes the directional derivative of the scalar function s with respect to the vector field $f + gu$ at point x. That is to say, the change rate of the scalar surface coordinate function $s(x)$, measured in the direction of the controlled field, is such that a crossing of the surface is guaranteed.

These conditions are equivalent to

$$\begin{cases} \lim_{s\to 0^+} \big\langle \mathrm{grad} s,\ f + gu^+ \big\rangle < 0, \\ \lim_{s\to 0^-} \big\langle \mathrm{grad} s,\ f + gu^- \big\rangle > 0. \end{cases} \tag{13}$$

The geometrical meaning is that on Σ the projections of the controlling vector fields $f + gu^+$ and $f + gu^-$ on (grad$s$) are of the opposite sign, and hence, the controlled fields locally point towards the surface Σ (figure 3).

In practice, sliding motion is not attainable; imperfections such as hysteresis, delays, sampling and unmodelled dynamics will result in a chattering motion in a neighbourhood of the sliding surface, as it has been schematized in figure 4. Such a real model will usually lie in the field of ordinary differential equations, and therefore, there is no need for Filipov's theory.

Moreover, if the control functions u^+ and u^- can be designed arbitrarily, the next proposition gives a very simple condition for Σ to be a sliding surface.

Proposition 2. A necessary and sufficient condition for the existence of control functions u^+ and u^- making Σ be a sliding surface is

$$\langle \mathrm{grad} s, g\rangle \neq 0, \tag{14}$$

which is known as the *transversality condition*.

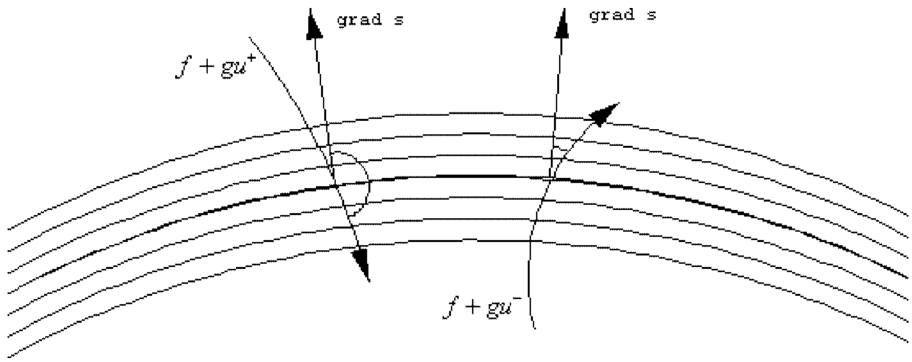

Figure 3. A graphic interpretation of the conditions given in (13).

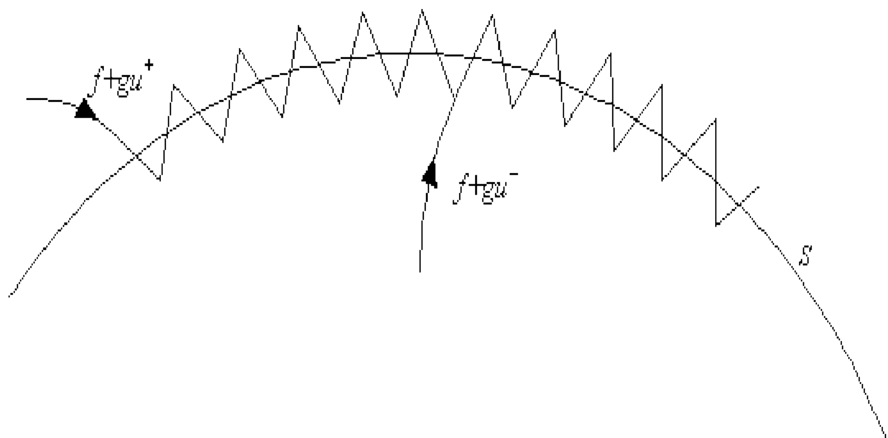

Figure 4. Chattering.

The proof is easy and can be found in [1] where this subject is widely considered. The cornerstone of the proof is to take the function s^2 as a Lyapunov function; so would that be applied in the remark and in the former proposition.

As outline of the proof: since $\langle \mathrm{grad} s, g\rangle \neq 0$, equivalent control exists. Then taking

$$u = u_{\mathrm{eq}} - k\,\mathrm{sign}(s), \quad k > 0, \tag{15}$$

and assuming $\langle \mathrm{grad} s, g\rangle > 0$, $s^2(x) \geqslant 0$ qualifies as a Lyapunov function.

4. Problem solution

The control of the reactor is solved in this section by following steps in section 3.

4.1. Equilibrium points

According to definition 1, the equilibrium points of system (3)–(4) are the roots of the equations

$$0 = -C\omega + C(1-N)e^{N/\gamma}, \tag{16}$$

$$0 = -N\omega + C(1-N)e^{N/\gamma}\frac{1+\beta}{1+\beta-N}, \tag{17}$$

which have a trivial solution of no interest in $(C, N) = (0, 0)$. The case $\omega = 0$ in the original system provides two other solutions, namely, $(C, 1)$ and $(0, N)$. Assuming $C > 0$, $N > 0$ and $\omega \neq 0$, it can be proved that the system equilibrium points lie on the parabola

$$C = -\frac{1}{1+\beta}N^2 + N, \tag{18}$$

which has the vertex on $((1+\beta)/4, (1+\beta)/2)$ and cuts the axis N on $(0, 0)$ and $(0, 1+\beta)$.

4.2. Sliding mode control methodology

The sliding mode control design for the bioreactor follows the steps set out in section 3, namely:

(1) the design of a sliding surface that guarantees that a stable equilibrium point will be reached, and

(2) the design of an appropriate control law taking into account (15) and the constraints of ω derived from the physical model.

In the case under discussion, the phase space is the plane and the sliding surface is reduced to a line.

4.2.1. Design of the sliding surface

(1) Sliding surface. Let us select a line cutting the parabola at an equilibrium point previously stated. In order to simplify the solution, let us take the line as a straight line $AC + BN = D$. To verify the transversality condition (14), $D \neq 0$ must be true, i.e., the straight line must not cross the origin (0,0). There is no loss of generality in assuming $D = 1$. Some brief remarks on the equilibrium points stability will be considered in the last paragraph of this subsubsection.

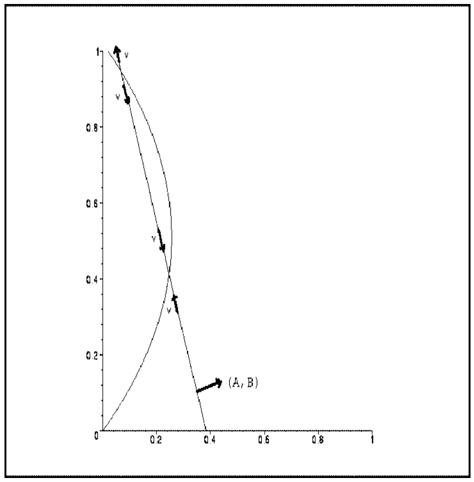

Figure 5. Stability of the equilibrium points when the line has negative slope.

(2) Equivalent control. The equivalent control ω_{eq} derived, as in (11), from

$$\frac{\mathrm{d}}{\mathrm{d}t}(AC + BN - 1) = 0$$

is given by

$$\omega_{\mathrm{eq}} = \frac{C(1-N)\mathrm{e}^{N/\gamma}}{AC + BN}\left(A + B\frac{1+\beta}{1+\beta-N}\right). \tag{19}$$

On the sliding line Σ equation (19) results in

$$\omega_{\mathrm{eq}}|_{\Sigma} = C(1-N)\mathrm{e}^{N/\gamma}\left(A + B\frac{1+\beta}{1+\beta-N}\right).$$

(3) Ideal sliding dynamics. Once $\omega = \omega_{\mathrm{eq}}$, the sliding line is dynamic-invariant. The ideal sliding dynamics, that is, the dynamics resulting from the restriction to the sliding line, are given by the vector field

$$\vec{v} = C(1-N)\mathrm{e}^{N/\gamma}\left(C\frac{1+\beta}{1+\beta-N} - N\right)(-B, A). \tag{20}$$

The dynamics are characterized by the stability of the two equilibrium points obtained from the intersection of the parabola and the sliding line, one of those being stable and the other unstable. In figures 5 and 6, the ideal sliding dynamics close to the equilibrium points have been depicted. From (20), stability of the equilibrium point clearly depends on $(-B, A)$ and the sign of $C(1+\beta)/(1+\beta-N) - N$ (positive or negative depending on which side of the parabola is considered). The straight lines with a neg-

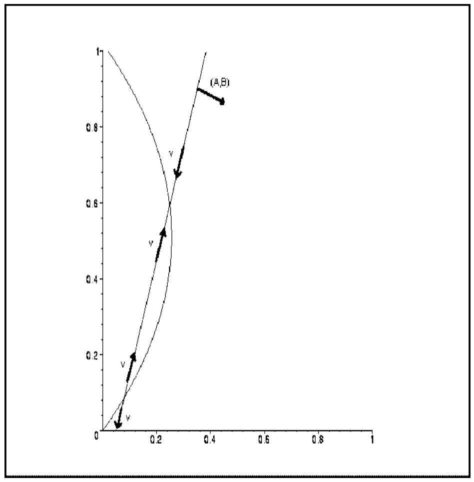

Figure 6. Stability of the equilibrium points when the line has positive slope.

ative slope $AC + BN = 1$, $A, B > 0$, cut the parabola at two intersection points: the lower stable and the upper unstable. On the other hand, for straight lines with a positive slope $AC - BN = 1$, $A, B > 0$, the lower equilibrium point is unstable and the upper stable.

Thus, in order to have a stable equilibrium point as ideal sliding dynamics, let us take $AC + BN = 1$, $A, B > 0$, which cuts the parabola at two intersection points: the lower stable, inside the square and the upper unstable, outside.

4.2.2. Control policy

To define the control strategy, (15) is taken into account. In addition,

(1) the input w must belong to the interval $[0, 2]$;

(2) the square $[0, 1] \times [0, 1]$ should be dynamic-invariant.

As in (15), let us now take $\omega = \omega_{eq} + \widehat{\omega}$ and look for conditions for $\widehat{\omega}$ such that $0.5(AC + BN - 1)^2$ is a Lyapunov function for the system. A straightforward computation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}(AC + BN - 1)^2 = -(AC + BN - 1)(AC + BN)\widehat{\omega} < 0. \tag{21}$$

As for the dynamic-invariance of the square $[0, 1] \times [0, 1]$, note that

1. On $N = 0$, the dynamic vector field is $(\dot{C}, \dot{N}) = C(1 - \omega, 1)$. The vertical component is positive, so trajectories go up.
2. On $N = 1$, the dynamic vector field is $(\dot{C}, \dot{N}) = \omega(-C, -1)$. The vertical component is negative, so trajectories go down.
3. On $C = 0$, the dynamic vector field is $(\dot{C}, \dot{N}) = (0, -N\omega)$. The horizontal component is zero, so trajectories do not go to the left-hand side.

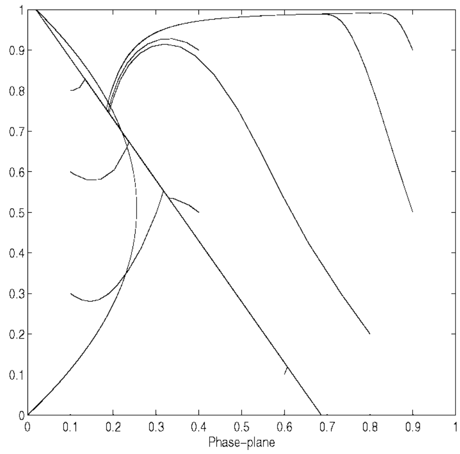

Figure 7. (C, N) phase-state diagram with different trajectories controlled by sliding mode.

4. On $C = 1$, the dynamic vector field is

$$(\dot{C}, \dot{N}) = \left(-\omega + (1 - N)\mathrm{e}^{N/\gamma}, -N\omega + (1 - N)\mathrm{e}^{N/\gamma}\frac{1+\beta}{1+\beta-N}\right).$$

If $\omega > (1 - N)\mathrm{e}^{N/\gamma}$, trajectories go to the left-hand side.

Taking into account all of the previous considerations, the control policy is defined by

$$\begin{cases} \omega = \omega_{\mathrm{eq}} + \widehat{\omega} & \text{if } 0 \leqslant \omega_{\mathrm{eq}} + \widehat{\omega} \leqslant 2, \\ \omega = 0 & \text{if } \omega_{\mathrm{eq}} \leqslant 0, \\ \omega = 2 & \text{if } 2 \leqslant \omega_{\mathrm{eq}}, \end{cases} \tag{22}$$

where, in turn, $\widehat{\omega}$ should satisfy

$$\begin{cases} \widehat{\omega} < 0 & \text{if } AC + BN < 1, \\ \widehat{\omega} = 0 & \text{if } AC + BN = 1, \\ \widehat{\omega} > 0 & \text{if } AC + BN > 1. \end{cases} \tag{23}$$

Notice that $AC + BN \geqslant 0$ in $[0, 1] \times [0, 1]$. Thus, in agreement with the previous equations, let us define $\widehat{\omega} = -\omega_{\mathrm{eq}}C$, where $\widehat{\omega} < 0$, and $\widehat{\omega} = (2 - \omega_{\mathrm{eq}})N$, where $\widehat{\omega} > 0$, resulting in a flow rate ω equal to

$$\omega = \begin{cases} \omega_{\mathrm{eq}}(1 - C) & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } AC + BN < 1, \\ \omega_{\mathrm{eq}} & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } AC + BN = 1, \\ \omega_{\mathrm{eq}}(1 - N) + 2N & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } AC + BN > 1, \\ 0 & \text{if } \omega_{\mathrm{eq}} \leqslant 0, \\ 2 & \text{if } 2 \leqslant \omega_{\mathrm{eq}}. \end{cases} \tag{24}$$

Figure 7 shows the phase-state diagram for the system. Several trajectories corresponding to different initial conditions scattered all over the square have been depicted. As can be seen, simulation results are in accordance with the desired behaviour. In this case, the point $(0.2196, 0.70)$ has been selected as a destination point, and the straight line joining this point and the intersection between the parabola and $N = 1$ as the sliding line. The controlled trajectories move in the square until reaching the sliding line, and then evolve on it up to the equilibrium point, where they remain.

5. Discrete switched control

When implementing the control in the real world, it is necessary to consider a sampling interval and convert the continuous process into a discrete one. The variables are sampled at the end of each sample period; then the input ω is evaluated and kept for the whole period. Thus, the behaviour of the control presents some differences.

In the case of discrete sliding mode control, it is necessary to introduce some changes. For the bioreactor, for example, a narrow band constituted by two straight lines ($AC + BN = 1 \pm \varepsilon$), which are parallel to the sliding line ($AC + BN = 1$), has been considered. The process used generalizes the continuous mode control using ω_{eq} inside the band, not only on the sliding line. As can be seen in the next definition, a slight modification has also been considered whenever $1 - \varepsilon < AC + BN < 1 + \varepsilon$ in order to improve the behaviour of the trajectories and to avoid taking very narrow bands. Finally, the discrete control ω is defined as

$$\omega = \begin{cases} \omega_{\mathrm{eq}}(1 - C) & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } AC + BN < 1 - \varepsilon, \\ \omega_{\mathrm{eq}}(AC + BN) & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } 1 - \varepsilon \leqslant AC + BN \leqslant 1 + \varepsilon, \\ \omega_{\mathrm{eq}}(1 - N) + 2N & \text{if } 0 \leqslant \omega_{\mathrm{eq}} \leqslant 2 \text{ and } 1 + \varepsilon < AC + BN, \\ 0 & \text{if } \omega_{\mathrm{eq}} \leqslant 0, \\ 2 & \text{if } 2 \leqslant \omega_{\mathrm{eq}}. \end{cases} \tag{25}$$

When the straight line $AC + BN = 1$ is secant to the parabola of the equilibrium points, the band is symmetrical to the straight line. However, if a straight line tangent to the parabola were considered, the band should change into a semi-band in order to reach a point in the parabola. In this case, it is necessary to adopt the equations which appear in (24).

6. Conclusions

The simulations carried out show that

(a) In the discrete case, sliding mode control presents a good performance because ω changes slightly with respect to the continuous case.

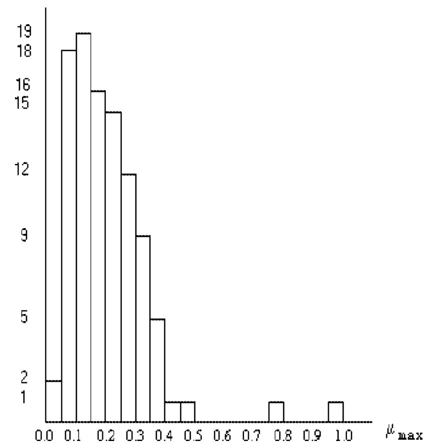

Figure 8. $\mu_{\max}$ values in the interval [0, 1] in percentage.

(b) Generally sliding mode control is a good option, but it is slow when the trajectory is on the sliding surface.

(c) The narrow band considered in discrete sliding mode control implies a reduction in precision. This reduction is less important than the tolerance margin accepted for the plant with which we are dealing.

Additionally, the time invested in measuring the concentrations of cells and nutrient has to be taken into account when defining the value of the sampling period. For the purposes of this study, a sample period of twenty minutes or half an hour is considered, which seems to be sufficient.

According to [8], the relationship between the real time t and the dimensionless time t^* which is used in computer simulations verifies (5), where $\mu(S_F)$ may be bounded by the maximum of $\mu(S)$; that is to say, $\mu_{\max}$. This value $\mu_{\max}$ appears tabulated in [8] for different kinds of cells and substratum. Figure 8 shows the percentage of $\mu_{\max}$ values in the interval [0,1]; the approximate average value is $\overline{\mu}_{\max} = 0.25$, all in hours^{-1} units.

If the $\mu_{\max}$ is 0.25 or has a relatively low value in the interval $[0, 1]$ (the most common situation (figure 8)), using a computer time $\Delta t^* = 0.125$ corresponding to half an hour of real time, the sliding mode is an appropriate control option. Nearly all of the cases are found in the interval $[0, 0.75]$. If $\mu_{\max}$ is 0.75 the computer time $\Delta t^* = 0.25$, which corresponds to twenty minutes of real time, may be used with excellent results.

Figures 9 and 10 corroborate the above comments. For the same cell concentration C, the system can be stabilized to two different nutrient concentrations N in the parabola. Namely, for $C = 0.2196$, one has $N = 0.32$ and $N = 0.70$. In figure 9, the destination point $(0.2196, 0.32)$ has been considered, $(0.2196, 0.70)$ being the destination point in figure 10. Two simulations corresponding to $\Delta t^* = 0.125$ and $\Delta t^* = 0.25$ are carried out in both cases. The resulting control input and phase-plane are plotted in the figures.

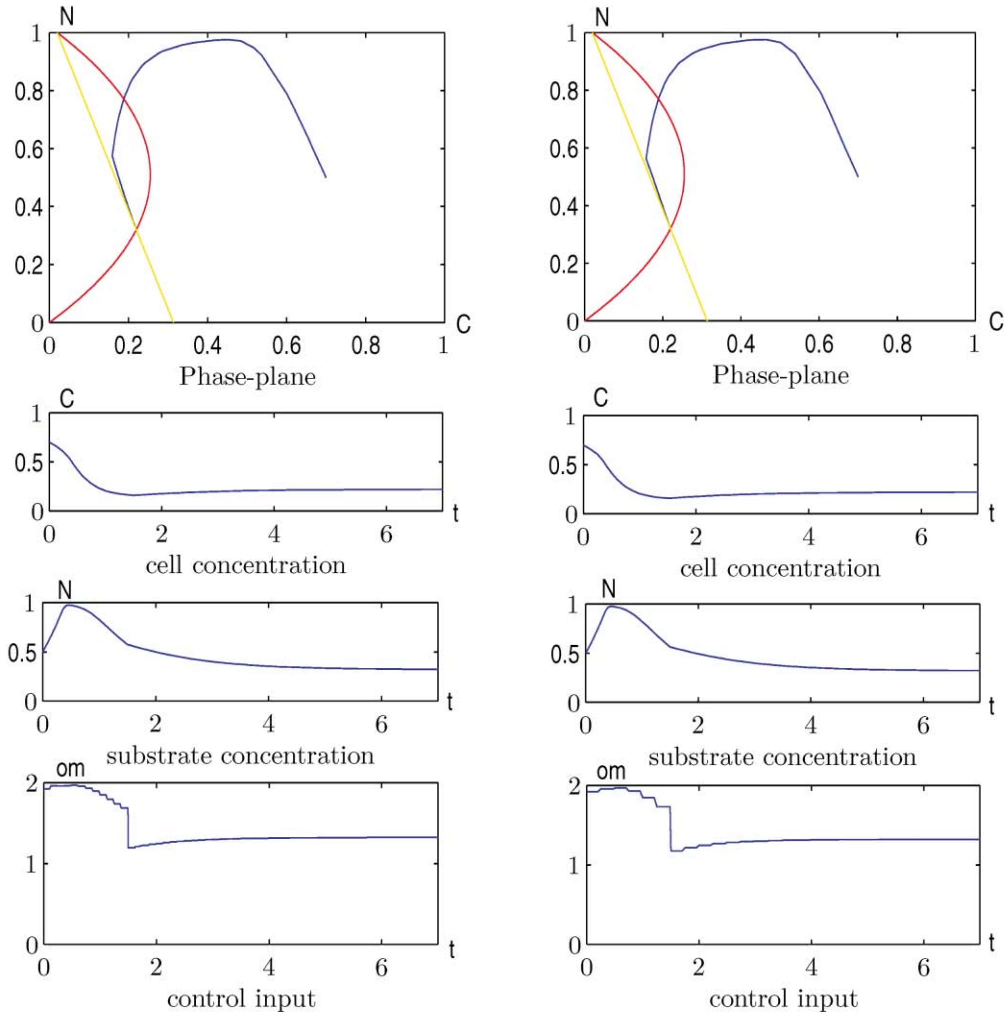

Figure 9. Discrete process of sliding mode control with $(0.2196, 0.32)$ destination point and $\Delta t^* = 0.125$ and $\Delta t^* = 0.25$.

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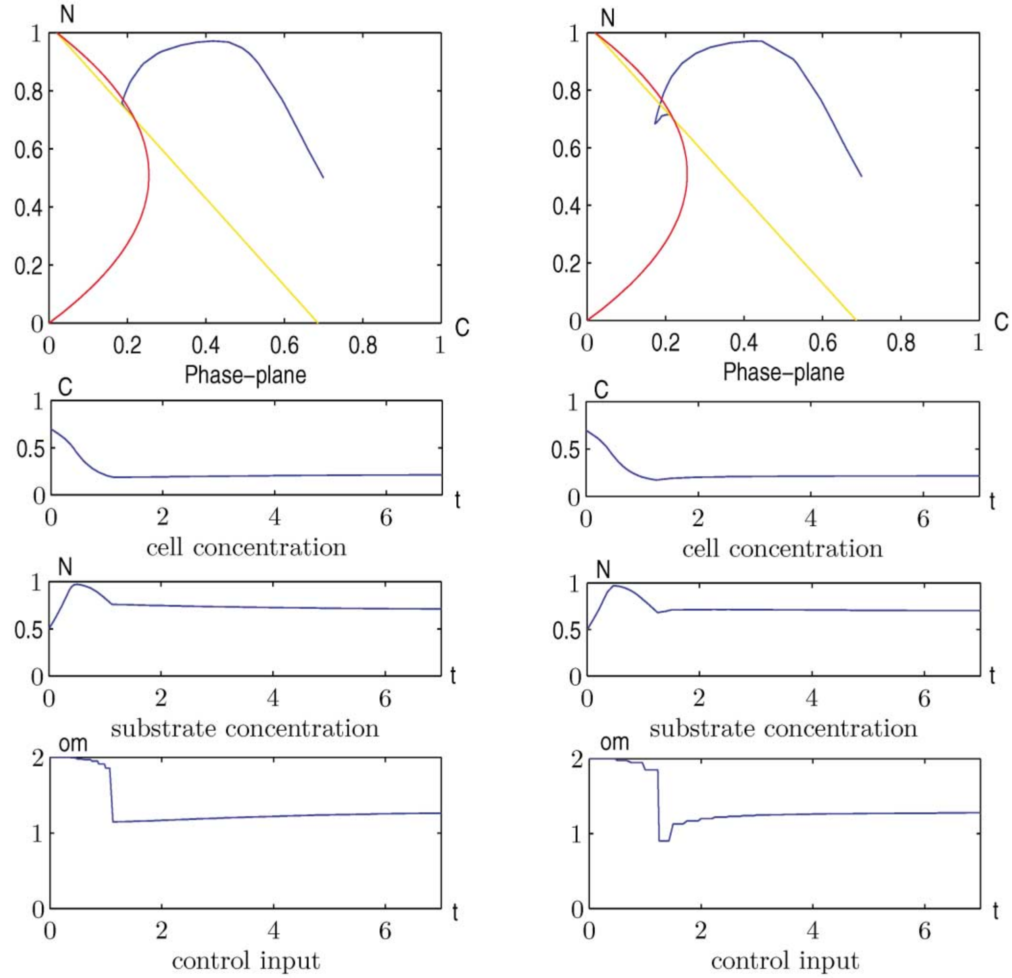

Figure 10. Discrete process of sliding mode control with (0.2196, 0.70) destination point and $\Delta t^* = 0.125$ and $\Delta t^* = 0.25$.

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